Celestial mechanics: Cheat sheet

Taavet Kalda: taavet.kalda@gmail.com *24.06.2022*

In what follows, results that are sparsely used are marked with a triangle symbol △.

Theoretical background

- **Newton's Law of Gravity:** The attractive force and potential energy between two bodies of masses M and m is given by

$$F = -G\frac{Mm}{r^2}, \quad U = -G\frac{Mm}{r}.$$

- **Kepler's 1st Law:** The orbit of every planet is an ellipse with the Sun at one of the foci.
  Remark. This also holds when the masses of the orbiting bodies are comparable, only that barycentre (centre of mass) is at the the focus.
- **Kepler's 2nd Law:** A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time, or equivalently, the **angular momentum** of a planet is conserved:

$$L = mv_\perp r = m\omega^2 r = \text{const}.$$

- **Kepler's 3rd Law:** The period T and semi-major axis a of an orbit are related as

$$\frac{T^2}{a^3} = \frac{4\pi^2}{G(M+m)},$$

  where M and ms are the masses of the two orbiting bodies.
  Remark. This also holds when the masses are comparable, only that a refers to the semi-major orbit of the motion of one body relative to the other. This simplifies to the semi-major axis of the planet when $m \ll M$.
- **Energy of an orbit:** The total energy of an elliptic orbit is given by its semi-major axis as

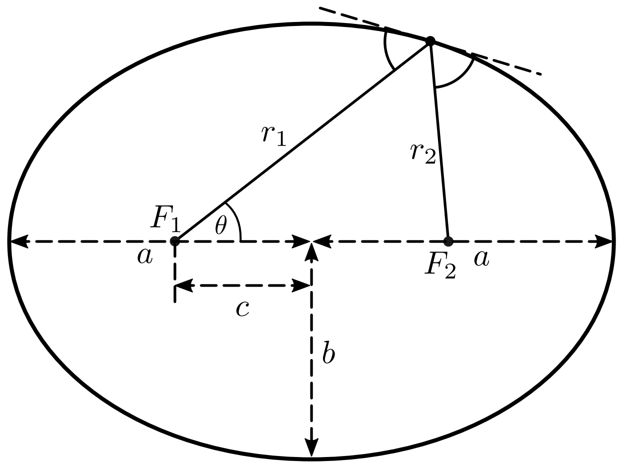
$$E = -\frac{GMm}{2a} = K + U = \frac{mv^2}{2} - G\frac{Mm}{r} = \text{const}.$$

Keplerian orbits

So far we've only considered closed orbits. As the energy of an orbiting body increases, it goes from closed ($E < 0$, ellipse) to parabolic ($E = 0$) to open ($E > 0$, hyperbolic). These are all conic sections, sometimes called Keplerian orbits.

Geometrically, their shapes are described with eccentricity e, showing how far from a circle they are ($e = 0$ for circle, $0 < e < 1$ for ellipse, $e = 1$ for parabola, $e > 1$ for hyperbola).

Ellipse



- An ellipse is defined by two foci such that any point on an ellipse satisfies $r_1 + r_2 = 2a$, where a is the semi-major axis (additionally, b is the semi-minor axis).
- Eccentricity is defined as

$$e = \frac{c}{a} = \sqrt{1 - \frac{b^2}{a^2}}.$$

  Exercise: Prove that these two equalities don't contradict each-other.
- The closest point to a foci is $r_{\min} = (1-e)a$ away and the farthest is $r_{\max} = (1+e)a$ away.
  Exercise: Make sure you understand how this follows from previous results.
- The angular momentum of a body on an elliptic orbit is given by $L = m\sqrt{GMa(1-e^2)}$.
  Exercise: Prove this!
  Remark. When the masses are comparable, m gets replaced by the reduced mass $\mu = (M^{-1} + m^{-1})^{-1}$.
- △In polar coordinates, the shape of an ellipse relative to a focus is given by

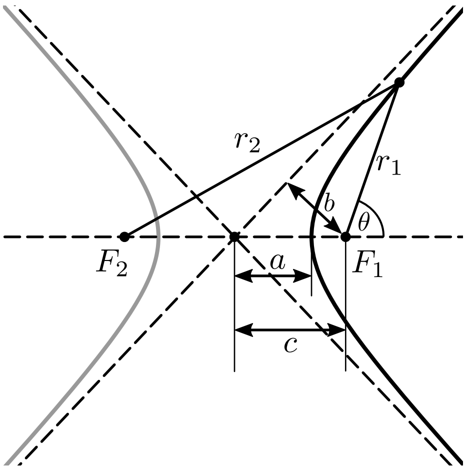
$$r(\theta) = \frac{a(1-e^2)}{1 - e\cos\theta}$$

- △An ellipse satisfies a reflective property in that the normal of the ellipse bisects the angle between $\vec{r}_1$ and $\vec{r}_2$ (see figure).

Parabola

- A parabola is effectively an ellipse where one of the foci is infinitely far away. Otherwise, all the formulas for the ellipse apply.
- As an orbit is only parabolic when the total energy takes on a single value of $E = 0$, Keplerian orbits in practice are never parabolic.

Hyperbola



- A hyperbola is defined by two foci such that any point on the hyperbola satisfies $r_2 - r_1 = 2a$, where a is the semi-major axis.
- Eccentricity is defined as

$$e = \frac{c}{a} = \sqrt{1 + \frac{b^2}{a^2}}.$$

  Exercise: Prove this!
- The closest point to a focus is $r_{\min} = (e-1)a$.
- The total energy of a hyperbolic orbit is

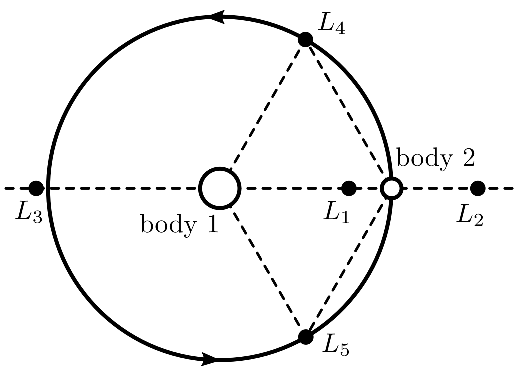
$$E = \frac{GMm}{2a}.$$

  Remark. Thus, for the intents of the elliptic orbital energy, one can imagine a hyperbolic orbit having a negative semi-major axis.
- △The polar equation with respect to a focus is

$$r(\theta) = \frac{a(e^2-1)}{1 - e\cos\theta} \qquad \text{for} \qquad -\frac{1}{e} < \cos\theta < \frac{1}{e}.$$

Lagrange points

Lagrange points are points of equilibrium under the influence of two massive orbiting bodies. The gravitational force from the two bodies alongside with the centrifugal force cancel out. L_1, L_2, and L_3 are unstable while L_4, and L_5 are stable.



General definitions

First cosmic velocity – Smallest necessary launch speed to get on a circular orbit.

Second cosmic velocity, aka escape velocity – Smallest necessary velocity to leave the gravitational influence of a stellar object

Pericenter and apocenter – The closest and farthest points in the orbit of a planetary body about its primary body respectively.

Perihelion and aphelion – The pericenter and apocenter of an orbit around the Sun.

Perigee and apogee – The pericenter and apocenter of an orbit around the Earth.